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Solitons with cubic and quintic nonlinearities modulated in space and time

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This work deals with soliton solutions of the nonlinear Schrödinger equation with cubic and quintic nonlinearities. We extend the procedure put forward in a recent paper [J. Belmonte-Beitia *et al.*, Phys. Rev. Lett. **100**, 164102 (2008)], and we solve the equation in the presence of a linear background and cubic and quintic interactions which are modulated in space and time. As a result, we show how a simple parameter can be used to generate brightlike or darklike localized nonlinear waves which oscillate in several distinct ways, driven by the space and time dependence of the parameters that control the trapping potential and the cubic and quintic nonlinearities.

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The nonlinear Schrödinger equation (NLSE) has been studied in a diversity of situations. The nonlinear interactions are usually of cubic nature, but there are systems which engender cubic and quintic (CQ) nonlinearities. The case of CQ nonlinearities opens new possibilities, and there are many interesting applications, especially in nonlinear optics, in fibers where the CQ nonlinearities may be used, for instance, to describe pulse propagation in double-doped optical fibers, when the type of dopant varies along the fiber, with the value and sign of the cubic and quintic parameters that control the nonlinearities being adjusted by properly choosing the characteristics of the two dopants [1]. There are other applications, and here we quote the Bose-Einstein condensates (BEC), where a diversity of nonlinearities may appear driven by controlled optical interactions [2].

A particularly important scenario for solitons concerns the NLSE in the case of a single spatial dimension with a linear term and cubic nonlinearity. However, we can change the linear parameter to cause it to trap the system. In this case it is named the Gross-Pitaevskii equation [3], and the linear parameter v is now modified to $v(x)$, with explicit dependence on the spatial coordinate x. Usually, the potential $v(x)$ is a background potential and has the form of the harmonic potential, with the purpose of trapping the system in a finite region in space, or some spatially periodically oscillating pattern, if the purpose is to entrap the system into a periodic lattice. This equation has gained further importance recently, mainly because of its direct application to the study of BECs [4], in fibers and in photonic crystals and other periodic systems [5,6]. Other applications include the study of nonlinear tunneling of spatial and temporal optical solitons in optical organic materials of practical use to ultrafast photonic technologies [7].

The pioneer work by Serkin and Hasegawa (SH) [8] has introduced an interesting procedure to deal with the nonautonomous NLSE. It is based on a similarity transformation, which transforms the nonautonomous NLSE equation into a stationary equation, which is easier to solve. The SH procedure has inspired several investigations [9]. In particular, in the recent work by Belmonte-Beitia, Pérez-García, Vekslerchik, and Konotop (BPVK) [10] one deals with the equation

$$i\frac{\partial\Psi}{\partial t} = -\frac{\partial^2\Psi}{\partial x^2} + v(x,t)\Psi + g(x,t)|\Psi|^2\Psi. \tag{1}$$

Here $v(x,t)$ and $g(x,t)$ may now vary, being functions with both space and time dependence, the first one being the trapping potential, and the second one describing the cubic nonlinearity. In the present work we shall follow the BPVK procedure, which is well explained in [10] and so we do not review it here. Instead, we extend the method to another problem, focusing our attention on the CQ nonlinear equation

$$i\frac{\partial\Psi}{\partial t} = -\frac{\partial^2\Psi}{\partial x^2} + v(x,t)\Psi + g_3(x,t)|\Psi|^2\Psi + g_5(x,t)\Psi^4\Psi, \tag{2}$$

where $v(x,t)$ is the trapping potential, and $g_3(x,t)$ and $g_5(x,t)$ control the nonlinear cubic and quintic interactions, respectively. The validity of the one-dimensional approach is given explicitly in [10], and the quintic term which we have added in (2) may simulate three-body collisions and/or deviation of the trapped condensate from the one dimensionality [11]. The trapping potential and nonlinearities to be used below are typical of BECs, and the results obtained may stimulate new experiments in the field.

In this work, we provide a way of making the BPVK procedure work in this scenario, involving the presence of CQ nonlinearities. The result allows one to obtain explicit solutions for some specific choices of parameters, leading us to interesting localized solutions of the bright or dark type, depending on the vanishing or not of the eigenvalue of the associated stationary nonlinear equation, as we explain below.

The idea is to write the solution of (2) as

$$\Psi(x,t) = \rho(x,t)e^{i\phi(x,t)}\Phi(\zeta(x,t)), \tag{3}$$

in order to rewrite (2) as the stationary equation

$$\mu\Phi(\zeta) = -\frac{d^2\Phi(\zeta)}{d\zeta^2} + G_3|\Phi(\zeta)|^2\Phi(\zeta) + G_5|\Phi(\zeta)|^4\Phi(\zeta), \tag{4}$$

where μ is the eigenvalue of the nonlinear equation above and G_3 and G_5 are real constants which control the CQ nonlinearities. The substitution of (3) into (2) leads to (4), but now we must have

$$\rho\frac{\partial\rho}{\partial t}+\frac{\partial[\rho^2(\partial\phi/\partial x)]}{\partial x}=0, \tag{5a}$$

$$\frac{\partial\zeta}{\partial t}+2\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}=0, \quad \frac{\partial[\rho^2(\partial\zeta/\partial x)]}{\partial x}=0. \tag{5b}$$

We can introduce a function $\xi(x,t)$ such that $\zeta(x,t)=F(\xi(x,t))$. In this case, we write $\xi(x,t)=\gamma(t)x+\delta(t)$. This choice is interesting because it allows us to determine the width of the localized solution in the form $1/\gamma(t)$ and its center-of-mass position as $-\delta(t)/\gamma(t)$. With Eqs. (5) we can obtain the new set of equations

$$\rho(x,t)=[\gamma/(\partial F/\partial\xi)]^{1/2}, \tag{6a}$$

$$\phi(x,t)=-\frac{(\partial\gamma/\partial t)}{4\gamma}x^2-\frac{(\partial\delta/\partial t)}{2\gamma}x+a(t), \tag{6b}$$

$$v(x,t)=\frac{1}{\rho}\frac{\partial^2\rho}{\partial x^2}-\frac{\partial\phi}{\partial t}-\left(\frac{\partial\phi}{\partial x}\right)^2-\frac{\mu\gamma^4}{\rho^4}, \tag{6c}$$

$$g_3(x,t)=G_3\gamma^4\rho^{-6}, \tag{6d}$$

$$g_5(x,t)=G_5G_3^{-1}\rho^{-2}g_3(x,t)=G_5\gamma^4\rho^{-8}, \tag{6e}$$

where $a(t)$ is an arbitrary function of time. The choice of $F(\xi)$ is to be done in such a way that we obtain finite-energy solutions of the CQ nonlinear equation.

This is the general procedure, and we note from (6e) that if we take the limit $G_5\to 0$, we obtain $g_5(x,t)\to 0$, and this leads us back to the case set forward in [10]. This result is robust. In fact, we have shown that the procedure can be extended to other nonlinearities: if in (2) we add the new term $g_7(x,t)|\Psi|^6\Psi$, etc., then in (4) we should add $G_7|\Phi|^6\Phi$, etc., and after (6e) we should include the new expression $g_7(x,t)=G_7G_5^{-1}\rho^{-2}g_5(x,t)=G_7G_3^{-1}\rho^{-4}g_3(x,t)=G_7\gamma^4\rho^{-10}$, and so on. We will deal with the more general case in another work [12], and below we focus attention on the important case of CQ nonlinearities.

Up to here, we have been very general. However, to illustrate the procedure with examples of interest, let us focus our attention on the case of specific nonlinearities. This is a nontrivial task, but we know that in the case of BECs with controlled optical interactions, we can have a diversity of nonlinearities [1–11]. Thus, in the present work we suppose that the cubic nonlinearity is given explicitly by

$$g_3(x,t)=\gamma e^{\xi^2/b^2}, \tag{7}$$

where b is a real parameter which controls the behavior of the cubic nonlinearity—we will use $b=8$ where required. With this choice we have

$$\rho(x,t)=G_3^{1/6}\gamma^{1/2}e^{-\xi^2/6b^2}. \tag{8}$$

The quintic nonlinearity is now controlled by

$$g_5(x,t)=G_5G_3^{-4/3}e^{4\xi^2/3b^2}, \tag{9}$$

and the potential given by Eq. (6c) must have the form

$$v(x,t)=\omega^2x^2+f_1x+f_2-\mu\gamma^2G_3^{-2/3}e^{2\xi^2/3b^2}, \tag{10}$$

and ω, f_1, and f_2, are time-dependent functions such that

$$\omega^2(t)=\gamma^4+\frac{1}{4\gamma}\frac{d^2\gamma}{dt^2}-\frac{1}{2\gamma^2}\left(\frac{d\gamma}{dt}\right)^2, \tag{11a}$$

$$f_1(t)=2\gamma^3\delta+\frac{1}{2\gamma}\frac{d^2\delta}{dt^2}-\frac{1}{\gamma^2}\frac{d\gamma}{dt}\frac{d\delta}{dt}, \tag{11b}$$

$$f_2(t)=\gamma^2+\gamma^2\delta^2-\frac{1}{4\gamma^2}\left(\frac{d\delta}{dt}\right)^2-\frac{da}{dt}. \tag{11c}$$

We define $\chi=1/\gamma$ in order to rewrite (11a) in the form of the Ermakov-Pinney equation [10,13]

$$\frac{d^2\chi}{dt^2}+4\omega^2(t)\chi=\frac{4}{\chi^3}. \tag{12}$$

This equation has a solution for γ in the form

$$\gamma(t)=[2y_1^2(t)+2y_2^2(t)/W^2]^{-1/2}. \tag{13}$$

Here W is the Wronskian of the two linearly independent solutions y_1 and y_2 of the Mathieu equation $(d^2y/dt^2)+4\omega^2(t)y=0$. See Ref. [13] for more details on this.

We now choose

$$\omega^2(t)=1+\varepsilon\cos(\omega_0t), \tag{14}$$

in order to obtain analytical solutions. The above results are general and can be used to investigate explicit examples, as we consider below. Before searching for explicit solutions, however, let us remark that investigations concerning specific criteria for the adiabaticity of nonlinear wave equations and of soliton solutions to such equations and issues related to parametric amplification of elementary excitations due to the periodic modulation of the trapping potential and nonlinearities were studied before in [14,15], respectively.

An important issue which appears in the case of CQ nonlinearities is that we can solve Eq. (4) for bright and dark solitons, depending on the values of the parameters that control the nonlinearities in a specific system. As the first example, let us consider the simpler case in which the stationary equation (4) has vanishing eigenvalue, $\mu=0$. In this case, if we choose the cubic and quintic parameters as $G_3=2$ and $G_5=-3$, we can write Eq. (4) in the form

$$\frac{d^2\Phi(\zeta)}{d\zeta^2}=2\Phi^3(\zeta)-3\Phi^5(\zeta). \tag{15}$$

The solution is given by $\Phi(\zeta)=1/\sqrt{1+\zeta^2}$. It has a bell-shaped form, and it is of the bright type, as we plot in Fig. 1(a). To have the correct boundary condition, we must set $\zeta\to\pm\infty$ as $x\to\pm\infty$, and so must choose $\zeta(x,t)=F(\xi(x,t))$ properly in order to make the solution behave according to the required boundary conditions. According to Eqs. (6), we choose $F(\xi)=G_3^{-1/3}\int d\xi e^{\xi^2/3b^2}$. We note that the bright solution is localized, but it is thicker than the hyperbolic secant, the stan-

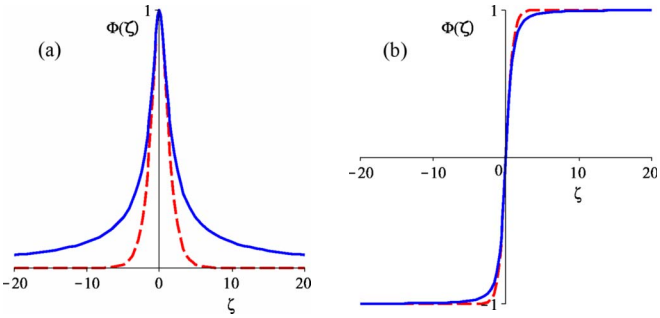

FIG. 1. (Color online) Plots of $\Phi(\zeta)$ for the bright soliton (a) (blue, solid line) and the sech(ζ) (red, dashed line) and for the dark soliton (b) (blue, solid line) and the tanh(ζ) (red, dashed line).

dard bright soliton, which is also plotted in Fig. 1(a), for comparison.

As we know, the form found above for the bright soliton can also be obtained in models of relativistic scalar fields, but this will be reported in another work [16]. We will get into this following the lines of [17], which suggests several distinct ways of modifying the bell-shaped form of brightlike solutions, an issue of direct interest for practical applications, as we will illustrate in [12].

To get to the darklike solution, let us now choose a nonvanishing eigenvalue, $\mu \neq 0$. In order to get explicitly simple analytical solution we consider, for instance, the case $\mu=3$, $G_3=6$, and $G_5=-3$. In this case, Eq. (4) changes to

$$\frac{d^2\Phi(\zeta)}{d\zeta^2} = -3\Phi(\zeta) + 6\Phi^3(\zeta) - 3\Phi^5(\zeta). \tag{16}$$

The solution is given by $\Phi(\zeta)=\zeta/\sqrt{1+\zeta^2}$. It has the form of a kink, as we show in Fig. 1(b), and so $|\Phi(\zeta)|^2$ is now of the dark type. It is interesting to note that this form of solution has also appeared in high-energy physics, as reported before in [18]. It is thicker than the standard darklike soliton which is described by the hyperbolic tangent, which we also plot in Fig. 1(b), for comparison.

Equation (15) leads to the brightlike solutions. In this case the wave function which solves Eq. (2) acquires the form

$$\Psi(x,t) = \frac{2^{1/6}\gamma^{1/2}e^{i\phi}e^{-\xi^2/6b^2}}{\sqrt{1+\zeta^2}}, \tag{17}$$

where $\phi=\phi(x,t)$ is real, obtained via the Eq. (6b).

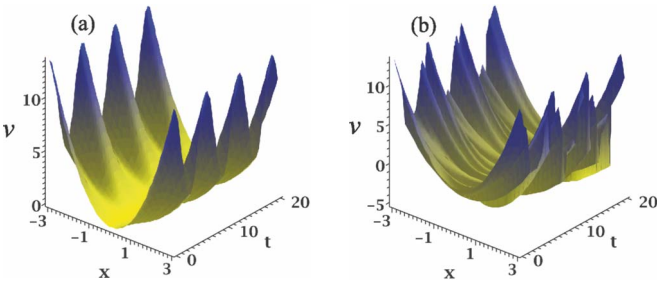

FIG. 2. (Color online) Plots of the trapping potential $v(x,t)$ given by (18a) for the resonant brightlike case (a) and by (20a) for the resonant darklike case (b), in the range $-3<x<3$ for the time evolution in the interval $0<t<20$.

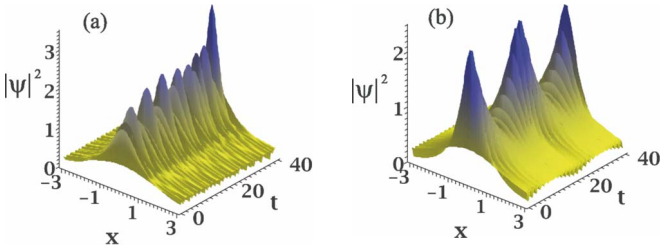

FIG. 3. (Color online) Plots of $|\Psi(x,t)|^2$ for the resonant bright soliton for $\epsilon=0.5$ and $\omega_0=1$ (a) and of the breathing bright soliton for $\epsilon=0$ (b). The initial data for Eq. (12) are $\chi(0)=\sqrt{2}$ and $d\chi/dt(0)=0$.

A specific solution can be constructed, which leads to resonant solitons. It is given by $\delta(t)=0$, $\mu=0$, and $a(t)=\int\gamma^2(t)dt$. In this case we obtain

$$v(x,t) = \omega^2(t)x^2, \tag{18a}$$

$$g_5(x,t) = -\tfrac{3}{2}2^{-1/3}e^{4\xi^2/3b^2}, \tag{18b}$$

with $g_3(x,t)$ given by (7). In Fig. 2(a) we plot the trapping potential (18a) in the standard form to show how it behaves as a function of space and time. This is to be compared with the trapping potential of the other case, for $\mu\neq 0$, where the dark solitons appear.

Solutions of the brightlike form nicely appear when one adequately chooses both ε and ω_0 in Eq. (14). In Fig. 3, we show the resonant and breathing solitons for specific choices of the parameters $\epsilon=0.5$, $\omega_0=1$, and $\epsilon=0$, respectively. We can also obtain quasiperiodic and moving solitons, but we will leave this to the longer work [12].

As we have already seen, the Eq. (16) gives the darklike solutions. In this case, Eq. (3) takes the form

$$\Psi(x,t) = \frac{6^{1/6}\gamma^{1/2}e^{i\phi}\zeta e^{-\xi^2/6b^2}}{\sqrt{1+\zeta^2}}. \tag{19}$$

If we take $\delta(t)=0$, $\mu\neq 0$, and $a(t)=\int\gamma^2(t)dt$, we obtain

$$v(x,t) = \omega^2(t)x^2 - \mu 6^{-2/3}\gamma^2 e^{2\xi^2/3b^2}, \tag{20a}$$

$$g_5(x,t) = -\tfrac{1}{2}6^{-1/3}e^{4\xi^2/3b^2}, \tag{20b}$$

with $g_3(x,t)$ given by (7). In Fig. 2(b) we plot the trapping potential (20a) in the standard form, for the resonant solution, to show how it behaves as a function of space and time.

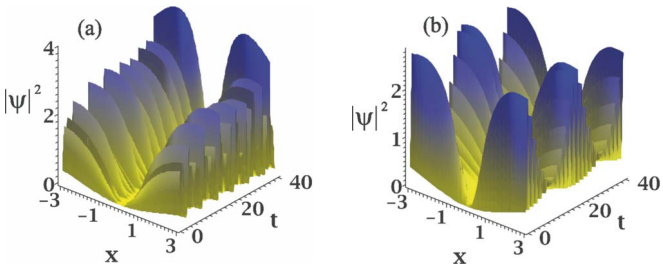

FIG. 4. (Color online) Plots of $|\Psi(x,t)|^2$ for the resonant dark soliton for $\epsilon=0.5$ and $\omega_0=1$ (a) and of the breathing dark soliton for $\epsilon=0$ (b). The initial data for (12) as in Fig. 3.

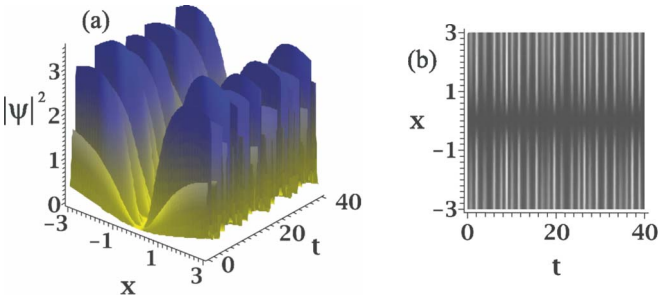

FIG. 5. (Color online) Plot of $|\Psi(x,t)|^2$ for the quasiperiodic dark soliton with $\varepsilon=0.5$ and $\omega_0=\sqrt{2}/2$ in (14) (a). The quasiperiodic behavior is better seen from the soliton profile at panel (b). The initial data for (12) as in Fig. 3.

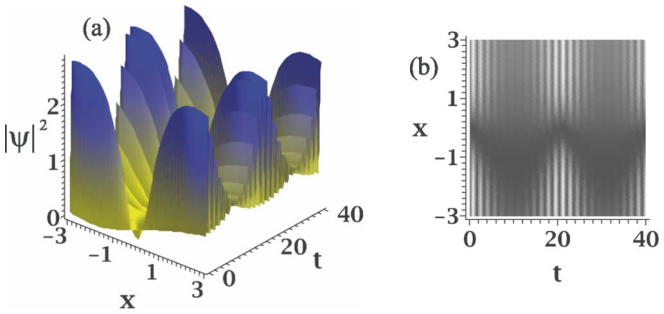

FIG. 6. (Color online) Plot of $|\Psi(x,t)|^2$ for the moving breathing dark soliton (a). The center-of-mass motion is better seen from the profile at panel (b). The initial data for (12) as in Fig. 3.

Here we see that for the darklike soliton, the trapping potential has a different behavior, if compared with the case of brightlike soliton—see Fig. 2. The presence of a nonvanishing μ modifies both the form and deepness of the trapping potential in the darklike case.

In the darklike case, we also have several soliton solutions. The resonant dark soliton appears with an appropriate choice for ε and ω_0 in Eq. (14), as we show in Fig. 4. We can also have the breathing soliton. It is obtained taking $\varepsilon=0$ in (14). In this case we have $\omega(t)=1$, and this leads us to $\gamma(t)=2/\sqrt{1+15\cos^2(2t)}$. The $|\Psi(x,t)|^2$ for darklike soliton of the breathing type is also shown in Fig. 4. If we take $\omega_0=\sqrt{2}/2$ in (14), we can use the above $\gamma(t)$ to obtain quasiperiodic solitons, as we show explicitly in Fig. 5.

Up to here we have chosen $\delta(t)=0$, to fix the center of mass of the solution; however, if we choose $\delta(t)\neq 0$ we can cause it to move. For simplicity, we consider the case $\varepsilon=0$ in (14), and take $\gamma(t)$ as above. Thus, we obtain $f_1(t)=0$ in (11b) and $f_2(t)=0$ in (11c) with the introduction of $\tau(t)=\int\gamma^2(t)dt$, which leads to $\delta(t)=\cos[2\tau(t)]$ and $a(t)=\tau(t)+(1/4)\sin[4\tau(t)]$. With this, we get to the case of a moving dark soliton, as we show explicitly in Fig. 6.

In this work we have extended the procedure set forward in the recent paper [10] to the case of cubic and quintic nonlinearities. As we have shown explicitly, the procedure is robust and can be generalized to higher (odd) nonlinearities very naturally, algorithmically. This result opens a new route for bright and dark solitons, with a diversity of possibilities of practical applications.

To strengthen the result, we have shown explicitly that the system can support bright or dark solitons, depending on the way we make the eigenvalue of the stationary nonlinear Eq. (4) vanish, in the case of bright solitons, or not, in the case of dark solitons. In the specific case of dark solitons, we have shown how to find the solutions with the resonant, breathing, quasiperiodic and moving behavior. A more detailed discussion is under preparation [12].

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